Algebraic geometry, complex analysis and combinatorics in spectral theory of periodic graph operators

Wencai Liu

Texas A&M University

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Periodic graphs

- $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$, $\mathcal{V}$: vertices, $\mathcal{E}$: edges
- Group: G
- Fundamental domain: $\mathcal{G}/G$ (finite).
- Invariants with respect to the group: $(u, v) \in \mathcal{E}$ implies $(gu, gv) \in \mathcal{E}$, $g \in G$

A simple $\mathbb{Z}^d$ graph

- $\mathcal{G} = \mathbb{Z}^d$
- Fix q_j, $j = 1, 2, \cdots, d$.
- Group: $G = q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$
- $W = \mathbb{Z}^d / G = \{n = (n_1, n_2, \cdots, n_d) : 1 \leq n_j \leq q_j\}$.

Periodic graph operators: discrete periodic Schrödinger operators

- Adjacency matrices + periodic potentials
- Let Δ be the discrete Laplacian on $\mathbb{Z}^d$: $u(n), n \in \mathbb{Z}^d$,

$$(\Delta u)(n) = \sum_{||n'-n||=1} u(n'),$$

where $||n|| = \sum_{i=1}^{d} |n_i|$ for $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$.

- Periodic potentials V: $V(n+g) = V(n)$ for all $n \in \mathbb{Z}^d$ and $g \in G$.
- The discrete periodic Schrödinger operator $H_0 = \Delta + V$:

$$(H_0 u)(n) = (\Delta u)(n) + V(n)u(n), n \in \mathbb{Z}^d.$$

An example: $d = 1$

- For a vector $u(n)$, $n \in \mathbb{Z}$,

$$(\Delta u)(n) = u(n+1) + u(n-1)$$

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- $V = \{V(n)\}_{n\in\mathbb{Z}}$ is the potential.
- q_1 periodic: $V(n+q_1) = V(n)$ for any $n \in \mathbb{Z}$
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$$\Delta + V = \begin{pmatrix} \ddots & \ddots & 0 & 0 & \cdots & 0 \\ \ddots & V_1 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & V_2 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & 0 & 1 & V_{q_1} & \ddots \\ 0 & \dots & 0 & 0 & \ddots & \ddots \end{pmatrix}$$

Floquet transform in 1D: $k \in [0,1]$

- $H_0 = \Delta + V \cong \oplus_{k\in[0,1]} D_V(k)$, where

$$D_V(k) = \begin{pmatrix} V_1 & 1 & 0 & 0 & e^{-2\pi ik} \\ 1 & V_2 & 1 & \ddots & 0 \\ 0 & 1 & V_3 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ e^{2\pi ik} & \dots & 0 & 1 & V_{q_1} \end{pmatrix}$$

$D_V(k)$ in one dimension

- Eigen-equation
$$(\Delta + V)u = \lambda u \tag{1}$$
Floquet-Bloch boundary condition
$$u(n + q_1) = e^{2\pi ik}u(n), n \in \mathbb{Z}. \tag{2}$$
- By writing out $H_0 = \Delta + V$ on $\{u(n)\}, n = 1, 2, \cdots, q_1$, we obtain $D_V(k)$.

Discrete Floquet transform

- Fundamental domain W:

$$W = \{n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d : 1 \leq n_j \leq q_j, j = 1, 2, \cdots, d\}.$$

- Cardinality of W: $Q = q_1 q_2 \cdots q_d$
- Eigen-equation

$$(\Delta + V)u = \lambda u \tag{3}$$

  Floquet-Bloch boundary condition

$$u(n + q_j \mathbf{e}_j) = e^{2\pi i k_j} u(n), j = 1, 2, \cdots, d. \tag{4}$$

- By writing out $H_0 = \Delta + V$ as acting on the Q dimensional space $\{u(n), n \in W\}$, (3) and (4) translate into the eigenvalue problem for a $Q \times Q$ matrix $D_V(k)$.

- $\Delta + V$ is unitary equivalent to $\bigoplus_{k\in\mathbb{T}^d} D_V(k)$, where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.
- Let $P_V(k,\lambda)=\det(D_V(k)-\lambda I)$ (characteristic function).
- Many problems related to periodic Schrödinger operators: study $P_V(k,\lambda)$

Algebraic Geometry

- Let $z_j = e^{2\pi i k_j}, j = 1, 2, \cdots, d$. $z = (z_1, z_2, \cdots, z_d)$ and $k = (k_1, k_2, \cdots, k_d)$.
- $\mathcal{D}_V(z) = D_V(k)$.
- $\mathcal{P}_V(z, \lambda) = \det(\mathcal{D}_V(z) - \lambda I)$.
- $\mathcal{P}_V(z, \lambda)$ is a Laurent polynomial of λ and $z_1, z_2, \cdots, z_d$.

Main results: irreducibility

Theorem 1 (L. GAFA 2022)

Let $d \geq 3$. Then for any $\lambda \in \mathbb{C}$, the Laurent polynomial $\mathcal{P}_V(z, \lambda)$ (as a function of z) is irreducible.

Theorem 2 (L. GAFA 2022)

Let $d = 2$. Then the Laurent polynomial $\mathcal{P}_V(z, \lambda)$ (as a function of z) is irreducible for any $\lambda \in \mathbb{C}$ except for $\lambda = [V]$. Moreover, if $\mathcal{P}_V(z, [V])$ is reducible, $\mathcal{P}_V(z, [V])$ has exactly two non-trivial irreducible factors (count multiplicity).

When $d = 2$, for a constant function V, $\mathcal{P}_V(z, [V])$ has exactly two irreducible components.

Theorem 3 (L. GAFA 2022)

The Laurent polynomial $\mathcal{P}_V(z, \lambda)$ (as a function of z and λ) is irreducible.

Proof of two conjectures

- Bloch variety: $B(V) = \{(k, \lambda) \in \mathbb{C}^{d+1} : P(k, \lambda) = 0\}$
- Conjecture 1: Bloch variety is irreducible (modulo periodicity)
- Fermi variety: $F_\lambda(V) = \{k \in \mathbb{C}^d : P(k, \lambda) = 0\}$
- Conjecture 2: Fermi varieties $F_\lambda(V)$ are irreducible (modulo periodicity) for all λ but finitely many λ.
- The two conjectures have been mentioned in many articles [Knörrer-Trubowitz 1990, Bättig-Knörrer-Trubowitz 1991, Bättig 1992, Kuchment-Vainberg 2000, Kuchment 2016]

Previous results

- $d = 2$, the Bloch variety $(\mathcal{P}_V(z,\lambda))$ is irreducible [Bättig 1988].
- $d = 2$, the Fermi variety is irreducible except for finitely many values of λ [Gieseker-Knörrer-Trubowitz 1993]
- $d = 3$, the Fermi variety is irreducible for every λ [Bättig 1992].
- Previous approaches: construction of toroidal and directional compactifications of Fermi and Bloch varieties.

Spectral theory of periodic Schrödinger operators

- Algebraic properties of $\mathcal{P}_V(z, \lambda)$ play a crucial role in the study of periodic Schrödinger operators
- Combinatorics
- Complex analysis

Examples

- embedded eigenvalues [Kuchment-Vainberg CPDE 2000], [Kuchment-Vainberg CMP 2006], [Shipman CMP 2014], [L. GAFA 2022]
- properties of spectral band functions [L. GAFA 2022]
- inverse problems: IDS [Gieseker-Knörrer-Trubotwitz 1993 Book], isospectrality [L. CPAM 2024] and Borg's Theorem [L. preprint 2023]
- quantum ergodicity [L. JDE 2022 and Mckenzie-Sabri CMP 2023]

One example: embedded eigenvalues

- Perturbed periodic Schrödinger operators (continuous):

$$\Delta + V + v,$$

where V is periodic and v satisfies some decaying properties.

- $$\sigma(\Delta + V) = \cup_m [a_m, b_m], \sigma_p(\Delta + V) = \emptyset.$$

- $$\sigma_p(\Delta + V + v) \cap (\cup_m [a_m, b_m]) = \emptyset?$$

Applications: embedded eigenvalues

Perturbed periodic operators:

$$H = \Delta + V + v, \tag{5}$$

where V is a real periodic potential and v is a decaying function on $\mathbb{Z}^d$.
Spectral bands:

$$\sigma(\Delta + V) = \bigcup [a_m, b_m], \sigma_p(\Delta + V) = \emptyset.$$

Theorem 4 (L. GAFA 2022)

If there exist constants $C > 0$ and $\gamma > 1$ such that

$$|v(n)| \leq Ce^{-|n|^\gamma}, \tag{6}$$

then $H = \Delta + V + v$ does not have any embedded eigenvalues, i.e., for any $\lambda \in \bigcup (a_m, b_m)$, λ is not an eigenvalue of H.

Special case

Corollary 5

Assume $|v(n)| \leq Ce^{-|n|^\gamma}$ *for some* $C > 0$ *and* $\gamma > 1$*. Then* $\sigma_p(\Delta + v) \cap (-2d, 2d) = \emptyset$ *(no embedded eigenvalues).*

- $\sigma(\Delta) = [-2d, 2d]$.
- Compactly support v: [Isozaki-Morioka 2014]

Proof of Corollary 5

- Eigen-equation

$$(\Delta u)(n) + v(n)u(n) = \lambda u(n), n \in \mathbb{Z}^d.$$

- Prove by contradiction: $\lambda \in (-2d, 2d)$ and $u \in \ell^2(\mathbb{Z}^d)$

By the Fourier transform, one has that

$$h_0(x)u(x) + \psi(x) = \lambda u(x). \tag{7}$$

- $$h_0(x) = 2\sum_{j=1}^{d} \cos 2\pi x_j$$

- $$\psi(x) = \sum_{n\in\mathbb{Z}^d} v(n)u(n)e^{-2\pi i n\cdot x}$$

- $$u(x) = \sum_{n\in\mathbb{Z}^d} u(n)e^{-2\pi i n\cdot x}$$

Proof of Corollary 5

- $u \in L^2(\mathbb{T}^d)$, $\psi(x)$ is an entire function with order $\frac{\gamma}{\gamma-1} + \varepsilon$.
- $$u(x) = \frac{\psi(x)}{(h_0(x) - \lambda)}. \tag{8}$$
- Claim: $u(x)$ is an entire function with order $\frac{\gamma}{\gamma-1} + \varepsilon$.
- Then $|u(n)| \leq Ce^{-|n|^{\gamma-\varepsilon}}$ which contradicts the unique continuation result.

What do we need for perturbed periodic operators?

- Unique continuation results (standard arguments)
- Irreducibility of Fermi varieties ($\{x \in \mathbb{C}^d : h_0(x) - \lambda = 0\}$)
- Real Fermi varieties have dimension $d - 1$ (standard arguments)

Proofs of irreducibility

- Fix $\lambda \in \mathbb{C}$. Assume $\mathcal{P}(z, \lambda) = \prod_j f_j(z)$.
- Letting $(z_1, z_2, \cdots, z_{d-1}) \to (0, 0, \cdots, 0)$ in a proper way and solving $\mathcal{P}(z, \lambda) = 0$, one has either $z_d \to 0$ or $z_d \to \infty$.
- For any j, $f_j(z)$ meets either $z_1 = 0, z_2 = 0, \cdots, z_{d-1} = 0, z_d = 0$ or $z_1 = 0, \cdots, z_{d-1} = 0, z_d = \infty$.

Proofs: asymptotics/tangent cones after changing variables

- Define "asymptotics" of $\mathcal{P}(z,\lambda)$ at $(0,0,\cdots,0)$ and $(0,\cdots,0,\infty)$.
- The asymptotics (independent of potentials V) can be calculated explicitly.
- Both asymptotics are irreducible.
- For any fixed λ, $\mathcal{P}(z,\lambda)$ has at most two irreducible factors.
- Asymptotics: the lowest degree component of $\mathcal{P}_V(z_1^{q_1},\cdots,z_d^{q_d},\lambda)$.
- Changing variables: $z_j \to z_j^{q_j}$, $j=1,2,\cdots d$.
- The highest degrees of $\mathcal{P}_V(z,\lambda)$: $z_1^{\pm\frac{Q}{q_1}}, z_2^{\pm\frac{Q}{q_2}}, \cdots, z_d^{\pm\frac{Q}{q_d}}$.

Proofs: degree arguments

- $\mathcal{P}(z, \lambda)$ has at most two irreducible factors.
- Assume $\mathcal{P}(z, \lambda)$ has two factors $f_1(z)$ and $f_2(z)$. We are going to show that $d = 2$ and $\lambda = [V]$.
- $f_1(z)$ meets $(0, 0, \cdots, 0)$ and $f_2(z)$ meets $(0, \cdots, 0, \infty)$.
- $f_j(z) =$ the asymptotics $+$ higher degree terms, $j = 1, 2$.
- Difficulties: $z_d \to \frac{1}{z_d}$.
- Polynomials: $\mathcal{P}_1(z, \lambda) = z^w \mathcal{P}(z, \lambda)$.

Further developments: more general lattices/polynomials

- Fillman-L.-Matos JFA 2022
- Fillman-L.-Matos JFA 2024
- Faust-Lopez Garcia preprint 2023

Thank you