# Semi-classical limit in complex analysis

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# Semi-classical limit

- Semi-classical limit is the process about taking the limit of large quantum (or large orbits, energies).
- "Semi-classical limit" plays an important role in modern mathematical research.

# Plan of the talk

- Part I: Asymptotic behavior of Gaussian integrals: The asymptotic behavior of Gaussian integrals best represents the spirit of "semi-classical limit".
- Part II: Semi-classical limit in geometry: Witten introduced semi-classical analytic proof of classical Morse inequalities at 1982. After Witten, "semi-classical limit" becomes a powerful tool in modern mathematical research.
- Part III: Semi-classical limit in complex analysis: We will introduce a "semi-classical limit" proof of Fefferman's famous theorem about boundary behavior of biholomorphic mappings.

# Gaussian integrals

- Consider the integral $\int_{\mathbb{R}} e^{-x^2+ix^4}\chi(x)dx$, $\chi \in \mathcal{C}_c^\infty(D)$, $\chi \equiv 1$ near $x = 0$, $D$ is a small open set of 0 in $\mathbb{R}$.
- The study of such kind of integrals is closely related to many important problems in modern mathematical research.
- It is very difficult to calculate such kind of integrals.

# Semi-classical Gaussian integrals

- Consider the integral $I_k := \int_{\mathbb{R}} e^{k(-x^2+ix^4)}\chi(x)dx$, $k \gg 1$.
- Can we understand large $k$-behavior of $I_k$?

# Semi-classical Gaussian integrals

- $I_k := \int_{\mathbb{R}} e^{k(-x^2+ix^4)}\chi(x)dx = \frac{1}{\sqrt{k}}\int_{\mathbb{R}} e^{-x^2+ik(\frac{x}{\sqrt{k}})^4}\chi(\frac{x}{\sqrt{k}})dx.$
- $\lim_{k\to+\infty}\sqrt{k}I_k = \int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}.$
- Moreover, we can show that $I_k \sim k^{-\frac{1}{2}}\sqrt{\pi} + k^{-\frac{3}{2}}a_1 + k^{-\frac{5}{2}}a_2 + \cdots,\ a_j \in \mathbb{R}.$

# Stationary phase formula of Hörmander and Melin-Sjöstrand

- $D \Subset \mathbb{R}^n$: open set.
- $F \in \mathcal{C}^\infty(D)$, $\mathrm{Im}\, F \geq 0$.
- $\mathrm{Im}\, F(0) = 0$, $F'(0) = 0$, $\det F''(0) \neq 0$.
- $F' \neq 0$ in $D \setminus \{0\}$.

# Stationary phase formula of Hörmander and Melin-Sjöstrand

- Let $u \in \mathcal{C}_c^\infty(D)$.
- 

$$\int_{\mathbb{R}^n} e^{ikF(x)}u(x)dx = e^{ikF(0)}\det\left(\frac{kF''(0)}{2\pi i}\right)^{-\frac{1}{2}}\sum_{j<N} k^{-j}(L_j u)(0) + O(k^{-N}). \tag{1}$$

- $L_j$: Differential operator of order $\leq 2j$, $L_0 = I$.

# Stationary phase formula of Hörmander and Melin-Sjöstrand

- Let $u \in \mathcal{C}_c^\infty(D)$.
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- $L_j$: Differential operator of order $\leq 2j$, $L_0 = I$.
- The behavior of $u$ at the critical point determines the integral $\int_{\mathbb{R}^n} e^{ikF(x)}u(x)dx$.

# Example

- Let $\Gamma(\lambda + 1) = \int_0^{+\infty} e^{-t} t^{\lambda} dt$: Gamma function.
- Let $t = \lambda(1 + s)$.

# Example

- $\Gamma(\lambda+1) = \int_{-1}^{+\infty} e^{-\lambda(1+s)}\lambda^{\lambda+1}(1+s)^{\lambda}ds = e^{-\lambda}\lambda^{\lambda+1}\int_{-1}^{+\infty} e^{-\lambda(s-\log(1+s))}ds.$
- We now study the large $\lambda$-behavior of

$$e^{\lambda}\lambda^{-\lambda-1}\Gamma(\lambda+1) = \int_{-1}^{+\infty} e^{-\lambda(s-\log(1+s))}ds$$

(semi-classical limit of Gamma function).

# Example

- $$\begin{aligned}&\int_{-1}^{+\infty} e^{-\lambda(s-\log(1+s))}ds\\&=\int_{-1}^{+\infty} e^{-\lambda(s-\log(1+s))}u(s)ds+\int_{-1}^{+\infty} e^{-\lambda(s-\log(1+s))}(1-u(s))ds.\end{aligned}$$
- $u\in\mathcal{C}_c^\infty(U)$, $u\equiv 1$ near 0, $U$ is an open set of 0 in $\mathbb{R}$.

# Example

- Let $F(s) := s - \log(1+s)$.
- Since $F'(s) \neq 0$ outside $s = 0$, we can integrate by parts in $s$ and get

$$\int_{-1}^{+\infty} e^{-\lambda(s-\log(1+s))}(1-u(s))ds = O(\lambda^{-N}), \quad \forall N \in \mathbb{N}.$$

# Example

- From (1), we deduce

$$\int_{-1}^{+\infty} e^{-\lambda(s-\log(1+s))}u(s)ds = \sqrt{2\pi}\lambda^{-\frac{1}{2}}+a_1\lambda^{-\frac{3}{2}}+a_2\lambda^{-\frac{5}{2}}+\cdots .$$

- We get

$$\Gamma(\lambda+1) = (\frac{\lambda}{e})^{\lambda}\sqrt{2\pi\lambda}(1+b_1\lambda^{-1}+b_2\lambda^{-2}+\cdots).$$

# Example

- $\Gamma(n+1) = n!$, $n \in \mathbb{N}$.
- $\Gamma(n+1) = n! = (\frac{n}{e})^n \sqrt{2\pi n}(1 + O(n^{-1}))$.
- We obtain Stirling's formula.

# Semi-classical limit in geometry

- Witten used "semi-classical limit" to give a pure analytic proof of classical Morse inequalities.
- We will introduce briefly Witten's approach.

- $M$: compact smooth manifold of dimension $n$.
- Let $f \in \mathcal{C}^\infty(M)$.
- For $x_0 \in M$, we call $x_0$ a critical point (of $f$) if $(df)(x_0) = 0$.

# Morse theory

- For a critical point $x_0$, we say $x_0$ is non-degenerate if $\det\left(\frac{\partial^2 f}{\partial x_j \partial x_\ell}(x_0)\right)_{j,\ell=1}^{n} \neq 0$.
- $f$ is a Morse function if every critical point of $f$ is non-degenerate.
- We can always find a Morse function $f \in \mathcal{C}^\infty(M)$.

# Morse theory

- Fix a Morse function $f \in \mathcal{C}^\infty(M)$.
- $\mathrm{Crit}\,(f) := \{$all critical points of $f\}$.

# Morse theory

- Let $x_0 \in \mathrm{Crit}\,(f)$, assume that $\left(\frac{\partial^2 f}{\partial x_j \partial x_\ell}(x_0)\right)_{j,\ell=1}^{n}$ has $n_-$ negative eigenvalues, $n_+$ positive eigenvalues, $n_- + n_+ = n$.
- We call $n_-$ the index of $x_0$ and denote $\mathrm{ind}\,(x_0) = n_-$.
- For $j = 0, 1, \ldots, n$, let $\mathrm{Crit}\,(f, j) := \{x \in \mathrm{Crit}\,(f);\ \mathrm{ind}\,(x) = j\}$ and let $\sharp\mathrm{Crit}\,(f, j)$ denote the cardinal number of $\mathrm{Crit}\,(f, j)$.

# Morse theory

- $H^r(M) = \dfrac{\mathrm{Ker}\left(d: \mathcal{C}^\infty(M, \Lambda^r(T^*M)) \to \mathcal{C}^\infty(M, \Lambda^{r+1}(T^*M))\right)}{\mathrm{Im}\left(d: \mathcal{C}^\infty(M, \Lambda^{r-1}(T^*M)) \to \mathcal{C}^\infty(M, \Lambda^r(T^*M))\right)}$: $r$-th De Rham cohomology group.
- $\dim H^r(M)$: $r$-th Betti number of $M$.

**Theorem (Morse inequalities)**

- $\dim H^r(M) \leq \sharp\mathrm{Crit}\,(f,r)$, $r = 0, 1, \ldots, n$.
- $\sum_{r=0}^{q}(-1)^{q-r}\dim H^r(M) \leq \sum_{r=0}^{q}(-1)^{q-r}\sharp\mathrm{Crit}\,(f,r)$, $q = 0, 1, \ldots, n$.
- $\sum_{r=0}^{n}(-1)^{n-r}\dim H^r(M) = \sum_{r=0}^{n}(-1)^{n-r}\sharp\mathrm{Crit}\,(f,r)$.

- $\sum_{r=0}^{n}(-1)^r \dim H^r(M) = \chi(M)$: Euler characteristic number of $M$.

# Witten's idea

- For $k \in \mathbb{N}$, consider

$$\triangle_k^{(r)} := \left(d^* + k(df)^{\wedge,*}\right)\left(d + k(df)^{\wedge}\right) + \left(d + k(df)^{\wedge}\right)\left(d^* + k(df)^{\wedge,*}\right) : \mathcal{C}^\infty(M, \Lambda^r(T^*M)) \to \mathcal{C}^\infty(M, \Lambda^r(T^*M)).$$

- $e^{-\frac{t}{k}\triangle_k^{(r)}}(x,y) \in \mathcal{C}^\infty(M \times M, \Lambda^r(T^*M) \boxtimes (\Lambda^r(T^*M)^{-1})$: Heat kernel of $\triangle_k^{(r)}$.

# Witten's idea

**Theorem (Atiyah-Singer-Patodi-Witten)**

- *For every $t > 0$ and $k > 0$:*
- $\dim H^r(M) \leq \int_M \mathrm{Tr}\, e^{-\frac{t}{k}\triangle_k^{(r)}}(x,x)dV_M(x)$, $r = 0, 1, \ldots, n$
- $\sum_{r=0}^{q}(-1)^{q-r}\dim H^r(M) \leq \sum_{r=0}^{q}(-1)^{q-r}\int_M \mathrm{Tr}\, e^{-\frac{t}{k}\triangle_k^{(r)}}(x,x)dV_M(x)$, $q = 0, 1, \ldots, n$.
- $\sum_{r=0}^{n}(-1)^{n-r}\dim H^r(M) = \sum_{r=0}^{n}(-1)^{n-r}\int_M \mathrm{Tr}\, e^{-\frac{t}{k}\triangle_k^{(r)}}(x,x)dV_M(x)$.

# Semi-classical limit in geometry

- In general,it is very difficult to calculate $\mathrm{Tr}\, e^{-\frac{t}{k}\triangle_k^{(r)}}(x,x)$.
- When $k$ goes to infinity, $e^{-\frac{t}{k}\triangle_k^{(r)}}(x,y)$ converges to well-known heat kernel on $\mathbb{R}^n$ (Heat kernel of Harmonic oscillator).

- We can calculate that

$$\lim_{t \to +\infty} \lim_{k \to +\infty} \int_M \mathrm{Tr}\, e^{-\frac{t}{k}\triangle_k^{(r)}}(x,x) dV_M(x) = \sharp \mathrm{Crit}\,(f,r), \quad (2)$$

$r = 0, 1, \ldots, n$.

- From Theorem I and (2), we get classical Morse inequalities.

- Riemann mapping theorem: If $U$ is a non-empty simply connected open subset of $\mathbb{C}$, $U \neq \mathbb{C}$, then there exists a biholomorphic mapping $f$ from $U$ onto the open unit disk $D = \{z \in \mathbb{Z};\ |z| < 1\}$.
- The Riemann mapping theorem for high dimensional domains is not true.

- If the boundary of $U$ is smooth, then the Riemann mapping function can be extended smoothly to the boundary,
- On high dimensional case, it is important to know boundary regularities of a biholomorhic mapping.

- $M := \{z \in \mathbb{C}^n;\ \rho(z) < 0\}$, $\rho \in \mathcal{C}^\infty(\mathbb{C}^n, \mathbb{R})$, $d\rho \neq 0$ on $\partial M$.
- $M$ is called strongly pseudoconvex if $\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_\ell}(z)\right)_{j,\ell=1}^n$ is positive definite at every point of $\partial M := \{z \in \mathbb{C}^n;\ \rho(z) = 0\}$.

- Open problem in several complex variables: Let $M_1, M_2$ be bounded strongly strongly pseudoconvex domains in $\mathbb{C}^n$ and let $F : M_1 \rightarrow M_2$ be a biholormorphic map. If $F$ can be extended smoothly up to the boundary of $M_1$.
- Charles Fefferman solved this problem at 1974 (his Fields medal work).

- Fefferman used very difficult and complicated harmonic analysis to solve this problem.
- In the rest of this talk, I will give a "semi-classical limit" proof of Fefferman's theorem.

# Bergman kernels

- $M := \{z \in \mathbb{C}^n;\ \rho(z) < 0\}$: bounded strongly pseudoconvex domain.
- $H^0(M) := \left\{u \in L^2(M);\ \overline{\partial} u = 0\right\}$.

# Bergman kernels

- Let $\{f_1, f_2, \ldots\}$ be an orthonormal basis of $H^0(M)$.
- Bergman kernel $B(x,y) := \sum_{j=1}^{+\infty} f_j(x)\overline{f_j(y)}$.
- $B(x,y) \in \mathcal{C}^\infty(M \times M)$, $B(x,y) \in \mathcal{C}^\infty(M \times \overline{M})$.

# Bergman kernels

- The study of boundary behavior of $B(x, y)$ is closely related to many important problems in complex analysis and complex geometry.
- It is very difficult to study boundary behavior of $B(x, y)$.

# Bergman kernels

- The study of boundary behavior of $B(x,y)$ is closely related to many important problems in complex analysis and complex geometry.
- It is very difficult to study boundary behavior of $B(x,y)$.
- Fefferman:

$$B(x,x) = \frac{a(x)}{\rho(x)^{n+1}} + b(x)\log(-\rho(x)), \quad a, b \in \mathcal{C}^\infty(\overline{M}).$$

# Motivation

- Question: Can we have semi-classical version of $B(x, y)$?
- Goal: Find some kernel $A_k(x, y)$ such that $A_k$ can produce many holomorphic functions as $k \to +\infty$ and $A_k(x, y)$ is easier to calculate or use.

# Example

- $M = \mathbb{B}_n := \left\{ z \in \mathbb{C}^n ; \, |z|^2 < 1 \right\}$: unit ball.
- $B_{\mathbb{B}_n}(z, w) = \frac{n!}{\pi^n} \frac{1}{(1 - z\overline{w})^{n+1}}$.

# Weighted Bergman kernel on $\mathbb{B}_n$

- For $m \in \mathbb{N}$,
  $H^0_m(\mathbb{B}_n) = \operatorname{span}\{z_1^{\alpha_1}\cdots z_n^{\alpha_n}|_{\mathbb{B}_n};\ \alpha_1+\cdots+\alpha_n = m\}$.
- $B_m = B_m(x,y) = \sum_{j=1}^{d_m} g_{j,m}(x)\overline{g_{j,m}(y)}$, $\{g_{1,m},\ldots,g_{d_m,m}\}$: orthonormal basis of $H^0_m(\mathbb{B}_n)$.
- $B_m(x,y) \in \mathcal{C}^\infty(\overline{\mathbb{B}_n}\times\overline{\mathbb{B}_n})$.

# Weighted Bergman kernel on $\mathbb{B}_n$

- Let $\chi \in \mathcal{C}_c^\infty(I, \mathbb{R}_+)$, $\chi \equiv 1$ near $I_0$, $I_0 \Subset I \Subset \mathbb{R}_+$ open intervals.
- For $k \in \mathbb{N}$, put $A_k(z,w) := \sum_{m\in\mathbb{N}} \chi(\frac{m}{k}) B_m(z,w) \in \mathcal{C}^\infty(\overline{\mathbb{B}_n} \times \overline{\mathbb{B}_n})$.

- $A_k(z, w) = \frac{1}{\pi^n} \int_0^{+\infty} e^{ikt\Psi_0(z,w)} k^{n+1} \chi(t) t^n dt.$
- $\Psi_0(z, w) = i(1 - z\overline{w}).$
- Question: How to define $A_k(z, w)$ on general strongly pseudoconvex domains?

# Observation

- Let $R_0 = \sum_{j=1}^{n}(z_j \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial \overline{z}_j})$.
- $T_{R_0} := B_{\mathbb{B}_n} \circ R_0 \circ B_{\mathbb{B}_n} : \mathrm{Dom}\, T_{R_0} \subset L^2(\mathbb{B}_n) \to L^2(\mathbb{B}_n)$: Toeplitz operator on $\mathbb{B}_n$.
- $H^0_m(\mathbb{B}_n)$: eigenspace of $T_{R_0}$ corresponding to the eigenvalue $m$.

# Weighted Bergman kernel on strongly pseudoconvex domains

- $M := \{z \in \mathbb{C}^n;\ \rho(z) < 0\}$: bounded strongly pseudoconvex domain.
- $R := \sum_{j=1}^n \left(\frac{\partial\rho}{\partial\overline{z}_j}\frac{\partial}{\partial z_j} - \frac{\partial\rho}{\partial z_j}\frac{\partial}{\partial\overline{z}_j}\right)$.
- $T_R := \frac{1}{2}\Big(B \circ (R + R^*) \circ B\Big) : \mathrm{Dom}\, T_R \subset L^2(M) \to L^2(M)$: Toeplitz operator.

# Weighted Bergman kernel on strongly pseudoconvex domains

- $T_R$: self-adjoint.
- The spectrum $\mathrm{Spec}\,(T_R) \subset \mathbb{R}$ of $T_R$ consists only of isolated eigenvalues, is bounded from below and has only $+\infty$ as a point of accumulation.

# Weighted Bergman kernel on strongly pseudoconvex domains

- Let $\{f_1, f_2, \ldots\} \subset \mathcal{C}^\infty(\overline{M})$ be an orthonormal basis of $H^0(M)$ so that $T_R f_j = \lambda_j f_j$, $j = 1, 2, \ldots$.
- $A_k(x, y) := \sum_{j=1}^{+\infty} \chi(\frac{\lambda_j}{k}) f_j(x) \overline{f_j(y)} \in \mathcal{C}^\infty(\overline{M})$.
- The key point: When $k \to +\infty$, $A_k(x, y)$ converges to the unit ball case.

# Main result (jointly with George Marinescu)

## Theorem I

*With the notations and assumptions above, as $k \to +\infty$, on $\overline{M} \times \overline{M}$,*

$$
\begin{aligned}
&A_k(x,y) = \int_0^{+\infty} e^{ikt\Psi(x,y)} b(x,y,t,k)dt + O(k^{-\infty}), \\
&b(x,y,t,k) \sim \sum_{j=0}^{\infty} b_j(x,y,t)k^{n+1-j}, \quad b_0(x,y,t) \neq 0, \qquad (3)\\
&b_j(x,y,t), b(x,y,t,k) \in \mathcal{C}^\infty(\overline{M} \times \overline{M} \times I), \quad I \Subset \mathbb{R}_+,
\end{aligned}
$$

$$
\begin{aligned}
&\Psi(z,w) \in \mathcal{C}^\infty(\overline{M} \times \overline{M}), \quad \operatorname{Im}\Psi \geq 0, \\
&\Psi(z,z) = -i\rho(z). \qquad (4)
\end{aligned}
$$

# Remark

- $b_0(x, x, t) = \frac{1}{\pi^n} \det(\mathcal{L}_x)\chi(t)t^n \not\equiv 0$, $x \in \partial M$.
- By using the asymptotic expansion of $A_k(x, y)$, we can produce many holomorphic functions on $M$.

# Fefferman's theorem

**Theorem**

- *Let* $M_j$ *be a strongly pseudoconvex domain of* $\mathbb{C}^n$, $j = 1, 2$.
- *Let* $F : M_1 \to M_2$ *be a biholomorphic map.*
- *Then,* $F$ *extends smoothly to the boundary of* $M_1$.

- $$B_1(x,y) = B_2(F(x),F(y))\det F'(x)\overline{\det F'(y)}. \qquad (5)$$
- If $\det F'(x)$ is unbounded on $M_1$.
- $\exists\ \{x_s\}_{s=1}^{+\infty} \subset M_1$, $\lim_{s\to+\infty} x_s = p \in \partial M_1$, such that $\lim_{s\to+\infty} |\det F'(x_s)| = +\infty$.
- Assume $\lim_{s\to+\infty} F(x_s) = q \in \partial M_2$.

- From (3), we can find $a_0, a_1, \ldots, a_n \in M_2$ so that for every $k \gg 1$,

$$A_{k,2}(q, a_0) \neq 0, \tag{6}$$

$$\det\left(\frac{\partial}{\partial z_j}\left(\frac{A_{k,2}(x, a_\ell)}{A_{k,2}(x, a_0)}\right)\right)_{j,\ell=1}^{n}\Big|_{x=q} \neq 0. \tag{7}$$

- From (6), (7) and some straightforward calculation, we can find $a_0, a_1, \ldots, a_n \in M_2$ so that for every $k \gg 1$,

$$B_2(q, a_0) \neq 0, \tag{8}$$

$$\det\left(\frac{\partial}{\partial z_j}\Big(\frac{B_2(x, a_\ell)}{B_2(x, a_0)}\Big)\right)_{j,\ell=1}^{n}|_{x=q} \neq 0. \tag{9}$$

- $$B_1(x_s, F^{-1}(a_0)) \\ = B_2(F(x_s), a_0) \det F'(x_s) \overline{\det F'(F^{-1}(a_0))}. \tag{10}$$

- From (8) and (10), we get a contradiction.

- Similarly, $\det(F^{-1})'(x)$ is bounded and hence $\det F'|_{\partial M_1} \neq 0$.
- Define $u_j(z) := \frac{B_1(z,F^{-1}(a_j))}{B_1(z,F^{-1}(a_0))}$, $v_j(z) := \frac{B_2(z,a_j)}{B_2(z,a_0)}$, $j = 1, \ldots, n$.
- From (9), $\{v_1, \ldots, v_n\}$ is a system of local holomorphic coordinates defined near $q$.

- From (5), we have

$$\left(\frac{\partial}{\partial z_j}\Big(\frac{B_1(x,F^{-1}(a_\ell))}{B_1(x,F^{-1}(a_0))}\Big)\right)_{j,\ell=1}^{n}$$
$$=\left(\frac{\partial F_\ell}{\partial z_j}(x)\right)_{j,\ell=1}^{n}\circ\left(\frac{\partial}{\partial z_j}\Big(\frac{B_2(F(x),a_\ell)}{B_2(F(x),a_0)}\frac{\det F'(a_\ell)}{\det F'(a_0)}\Big)\right)_{j,\ell=1}^{n}. \tag{11}$$

- From (11), $\{u_1,\ldots,u_n\}$ is a system of local holomorphic coordinates defined near $p=F^{-1}(q)$.

# "Semi-classical limit" proof of Fefferman's theorem

- $v_j(F(z)) = \overline{\left(\frac{\det F'(F^{-1}(a_0))}{\det F'(F^{-1}(a_j))}\right)}\, u_j(z)$, $j = 1, \ldots, n$.
- In terms of local coordinates $\{u_1, \ldots, u_n\}$, $\{v_1, \ldots, v_n\}$,

$$F(u_1, \ldots, u_n) = \mathrm{diag}\left(\overline{\left(\frac{\det F'(F^{-1}(a_0))}{\det F'(F^{-1}(a_1))}\right)}, \ldots, \overline{\left(\frac{\det F'(F^{-1}(a_0))}{\det F'(F^{-1}(a_n))}\right)}\right).$$

- $F$ can be extended smooth to the boundary of $M_1$.

# Remark

- In a joint work with Marinescu, we establish a semi-classical Bergman asymptotic expansion on weakly pseudoconvex domains with $\mathbb{R}$-action.
- We establish Fefferman type regularity theorem about biholomoprphic mappings for weakly pseudoconvex domains with $\mathbb{R}$-action.

# Remark

- In a joint work with Marinescu, we establish a semi-classical Bergman asymptotic expansion on weakly pseudoconvex domains with $\mathbb{R}$-action.
- We establish Fefferman type regularity theorem about biholomoprphic mappings for weakly pseudoconvex domains with $\mathbb{R}$-action.
- Find "semi-classical limit" in your study.

Thank you for your attention!